The Remarkable Ibn al-Haytham
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The remarkable Ibn al-Haytham

JOHN D. SMITH

'I saw that I can reach the truth only through concepts whose matter are sensible things and whose form is rational.'

The achievements in experimental and theoretical science of the Arab scholar al-Haytham (also known as Alhazen, from his latinized first name al-Hasan) make him as much a figure of the sixteenth and seventeenth centuries as of his own tenth and eleventh centuries.

When his writings become known in the West the importance of his contribution to optics was widely recognized and he was studied by Galileo, Kepler, Fermat, Snell and Descartes. Mathematicians remember al-Haytham chiefly for *Alhazen's Problem* on the reflection of light from a circular mirror, which he solved by the method of conic sections; Huygens, Gregory, l'Hospital, Barrow (and many others) later took up the problem with the new analytical methods of geometry. Al-Haytham also wrote a commentary on the postulates of Euclid, and his attempted proof of the parallel postulate has similarities to Lambert's quadrilateral and Playfair's axiom in the eighteenth century. His theory of cognition may produce yet further interest in his work.

This essay includes a sketch of his life and main achievements in science; a summary of elementary problems in geometrical optics; a derivation by complex numbers of various solutions of Alhazen's Problem, including Huygens' famous construction; and further comments on optical problems.

A sketch of al-Haytham's life and scientific achievements

Al-Haytham was born about 965 and lived in Basra in present-day Iraq. According to tradition the eminent mathematician devised a way of controlling the flow of the Nile by building a dam across it south of Aswan. When the Caliph al-Hakim (996–1021) heard of the plan he invited him to Egypt to carry it out. However, on viewing the situation al-Haytham realized that the project was unfeasible, and from fear of the Caliph feigned madness to save his life. After the Caliph died al-Haytham resumed his teaching and writing, and he lived on in Cairo until about 1040.

In the flourishing Arab culture of the middle ages the classical tradition was both preserved and studied at a time when Europe had all but forgotten it. (Thus Western scholars of the late middle ages had their only access to many of the classical writings through contact with the Moslem world). The sources most influential for al-Haytham included those of Aristotle; the entire mathematical achievement of the Greeks, especially Euclid, Archimedes and Apollonius; and the writings of Ptolemy of Alexandria on astronomy and optics.

His greatest work is the Kitab al-Manazir or *Treasury of Optics*. Here we have the first modern description of physical light rays and their reception by the eye. This replaced a confusion of ideas which, in the later Greek period, had included the theory of 'visual rays' which leave the eye and sense luminous objects within the field of vision. Al-Haytham dismissed as preposterous the notion that rays extend outward from our eyes to the moon (for example). Instead, he said, luminous bodies emit light rays both directly into our eyes, and also onto secondary bodies which are illuminated themselves in various degrees, and whose light we receive in similar manner. In this way he accounted for the light of the moon.

He conducted a series of experiments on the rectilinear properties of light using a dark room with slits in an intermediate wall; demonstrated the laws of reflection of light by an accurate experimental method; and then built apparatus to investigate similar laws for refraction of light. He both believed in the importance of accurate experimentation and was a master of it. Though the conclusions from his experiments on refraction are not in error (within the ranges of his observations), he failed to discover the law which Descartes and Snell found 600 years later; nevertheless, his reasoning was similar to theirs, depending as it did on a parallelogram of velocities.

But how is the image of an object formed in our mind? From knowledge of the eye's interior he developed a theory of vision in which rays from an object enter the eye and are refracted through the crystalline humour (lens), to a point. According to al-Haytham, the crystalline humour is the organ which is sensitive to light rays and the information it acquires is transmitted to the brain via the optic nerve. Thus al-Haytham did not come to recognize (as Kepler did) the eye as a *camera obscura* in which rays pass through the lens to form an inverted image of the object on the sensitive retina, though he identified many of the key principles of vision and conducted experiments on the operation of the eye.

His contributions to astronomy are less original. Much of al-Haytham's work is a commentary on Ptolemy's *Almagest* and the complicated system of epicyclic gears by which Ptolemy modelled the motion of the solar system.

Alhazen's problem in the history of geometrical optics

Whilst the concept of mathematical rays and the rules of reflection may have been known since Plato's time, the more interesting properties date from the later period.

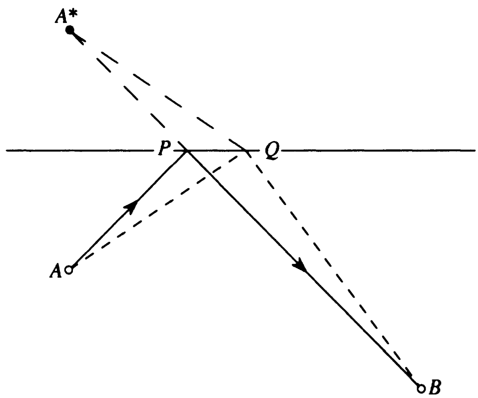

FIGURE 1.

If a ray passes from A to B via reflection at a point P on a plane mirror (figure 1) Heron of Alexandria recognized in the first century A.D. that the sum of the distances $|AP| + |PB|$ is a minimum: if A^* is the image of A in the plane, for any point Q in the plane $|AQ| + |QB| = |A^*Q| + |QB|$; this is least when A^*QB is a straight line. His explanation, 'nature does nothing in vain' was developed by Fermat into a general principle of least time.

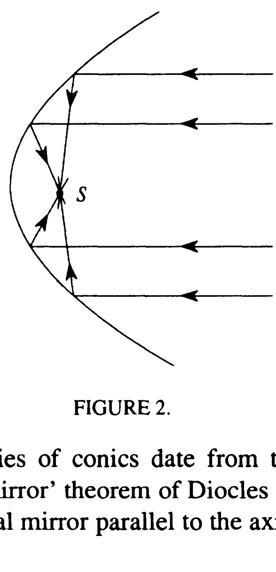

FIGURE 2.

The focussing properties of conics date from two or three centuries before. The 'burning mirror' theorem of Diocles demonstrates that rays which enter a paraboloidal mirror parallel to the axis are brought together at the point which Kepler later called the *focus* (figure 2); equivalently, rays which leave the focus are reflected into a beam parallel to the axis.

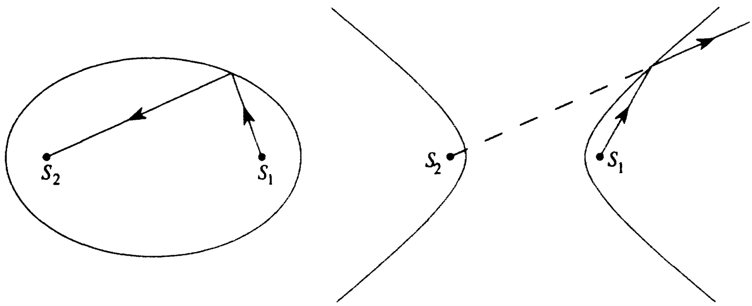

FIGURE 3.

For an ellipse, rays which leave one focus are brought together at the other focus after one reflection (figure 3); in the case of a hyperbola, the reflected rays appear to have come *from* the other focus. The sum of the focal distances $|PS_1| + |PS_2|$ is constant for the ellipse, and the *difference* of the focal distances is constant for the hyperbola.

Simple symmetrical problems on the reflection of light from a circular mirror were studied by Ptolemy, but it was al-Haytham who, in Book V of the *Kitab*, solved the general optical problem which bears his name:

> An object and an observer are at given positions in a plane; how do we locate the point or points on a circular mirror at which a ray is reflected from the object to the observer?

For his construction he used Apollonian methods of conic sections. He also solved the general three-dimensional problem for a cylindrical mirror, and the similar problem for a cone.

The problem began to interest mathematicians in the seventeenth century because it was amenable to the analytical methods which Descartes had introduced into geometry. Al-Haytham's solution is extremely long and complicated; the difficult Latin translation of the proof may never have been properly understood by these authors, although its form could have influenced them. The most important work on the problem after al-Haytham's is contained in the correspondence which Huygens and Sluse addressed to Oldenberg, secretary of the Royal Society, between 1669 and 1674. An infinite family of hyperbolae, parabolae and ellipses intersect the circle in the required points, and Huygens and Sluse competed with each other to find the most pleasing construction. Sluse's method uses a parabola, but no better solution has ever been found than the construction with a rectangular hyperbola which Huygens gave in 1672. We now obtain both of their solutions by complex numbers.

Analytical solution of Alhazen's Problem

In figure 4 a circle has centre O and any radius, and a ray passes from A to B after reflection at P. We represent A, B and P by the complex numbers a, b and z.

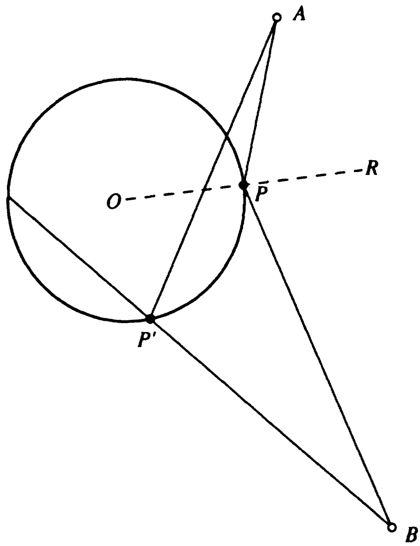

FIGURE 4.

The angles of reflection APR and RPB are equal to $\arg((a-z)/z)$ and $-\arg((b-z)/z)$ respectively, and by equating them we obtain

$$\arg\frac{(a-z)(b-z)}{z^2} = 0 .$$

The expression in square brackets is *real*, and therefore

$$\frac{(a-z)(b-z)}{z^2} = \frac{(\bar{a}-\bar{z})(\bar{b}-\bar{z})}{\bar{z}^2} \quad , \qquad (1)$$

where $\bar{z}$ is the complex conjugate of z.

Conversely equation (1) is satisfied either if angles APR and RPB are equal or if they differ by π . In figure 4 a ray which is reflected at P' appears to have come from B; thus P' also satisfies equation (1). We will call both kinds of points 'reflection points', and distinguish 'proper reflections' from 'backward reflections'. By rearrangement,

$$\bar{a}\bar{b}z^2 - ab\bar{z}^2 = [(\bar{a}+\bar{b})z-(a+b)\bar{z}]z\bar{z}\ .$$

If we take the coordinate axes along the angle-bisectors of OA and OB then ab is real, and by division

$$z^2 - \bar{z}^2 = [(a^*+b^*)z - \overline{(a^*+b^*)z}]z\bar{z}\ ,$$

where $a^* = 1/\bar{a}$ and $b^* = 1/\bar{b}$. Now write $c = \frac{1}{2}(a^*+b^*)$, so that

$$z^2 - \bar{z}^2 = 2(cz-\bar{c}\bar{z})z\bar{z} \tag{2}$$

If $c = |c|\,e^{i\delta}$ and $z = re^{i\theta}$ this defines a polar curve shown in figure 5 with equation

$$r = \frac{\sin 2\theta}{2|c|\sin(\theta+\delta)}\ ,$$

which Isaac Barrow derived in his Cambridge Lectures of 1669.

It is clear from equation (1) that the curve passes through A and B. By drawing circles with centre O we see how the number of reflexion points varies between two and four, depending on the radius of the mirror. For the circle drawn in figure 5 there are four intersections; P_2 and P_4 define proper reflections, and P_1 and P_3 backward reflexions. When A and B both lie inside the circle, graphs show that there are between two and four intersections; they are all proper reflection points.

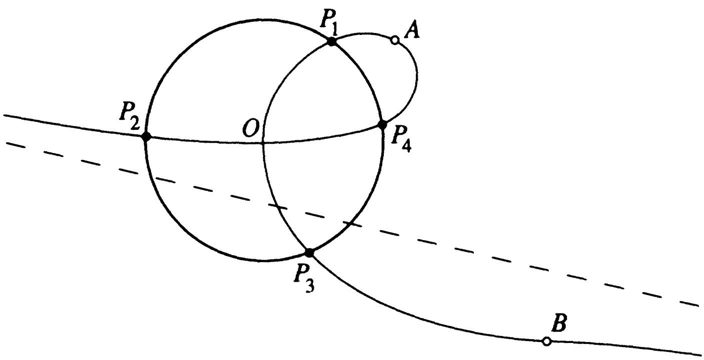

FIGURE 5.

With $c = c_1 + ic_2$ we obtain from (2) the cubic form

$$xy = (c_2x + c_1y)(x^2+y^2) \tag{3}$$

for Barrow's curve. Apart from degenerate cases when $c_1 = 0$ or $c_2 = 0$ (see below) the curve always has a loop which touches the coordinate axes at O, and an asymptote with equation

$$\frac{x}{c_1} + \frac{y}{c_2} = -\frac{1}{c_1^2 + c_2^2}.$$

We now assume, without any loss of generality, that the mirror is the unit circle $|z| = 1$. From (3), $xy = c_2x + c_1y$, and hence

$$(x - c_1)(y - c_2) = c_1c_2.$$

The reflection points therefore lie at the intersection of the circle with a rectangular hyperbola (figure 6). Now $c = (a^* + b^*)/2$ and a^* and b^* represent the inverses A^*, B^* of A, B in the circle. Therefore the centre $C(c_1, c_2)$ of the rectangular hyperbola is the mid-point of the inverse images of A and B in the mirror, the asymptotes are parallel to the angle-bisectors of OA and OB, and the hyperbola passes through the centre of the circle: this is the very elegant solution of Huygens. The hyperbola is the inverse image of Barrow's curve in the circle, and therefore passes through A^* and B^* as well !

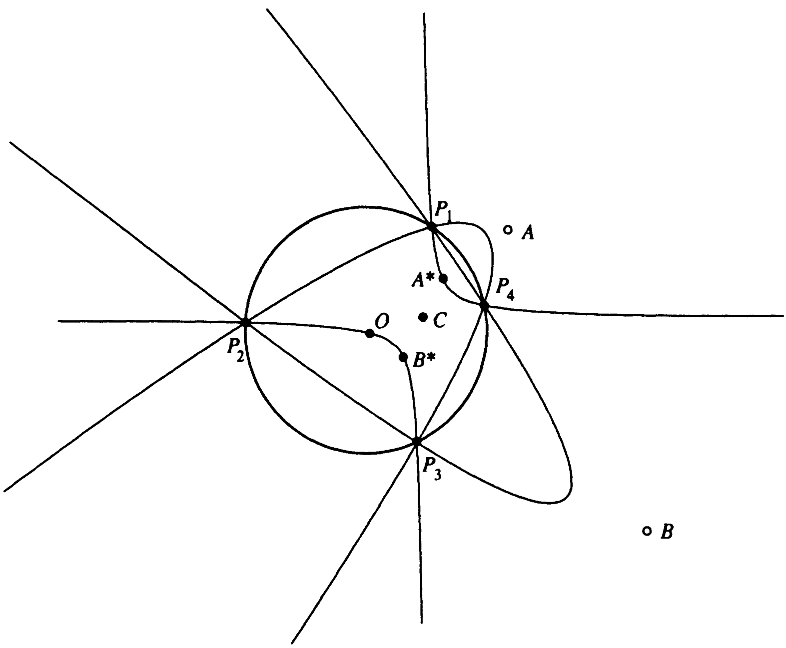

FIGURE 6.

The principal axes of a rectangular hyperbola are inclined at 45° to the asymptotes. If new coordinate axes lie in these directions, ab is *imaginary* and (2) becomes

$$Z^2 + \overline{Z}^2 = 2(CZ + \overline{CZ}) \ ,$$

where upper case letters refer to the new axes.
Hence

$$X^2 - Y^2 = 2(C_1X - C_2Y) \ .$$

The equation of any conic which passes through the common points of the rectangular hyperbola and the circle is a linear combination of (4) and $X^2 + Y^2 = 1$. By adding and subtracting we obtain two parabolae with perpendicular axes of symmetry:

$$C_1X + (Y - \tfrac{1}{2}C_2^2)^2 = \tfrac{1}{4}(C_2^2 + 2)$$

and

$$(X - \tfrac{1}{2}C_1^2)^2 + C_2Y = \tfrac{1}{4}(C_1^2 + 2) \ .$$

These are the parabolae in Sluse's solution. Their axes intersect at the midpoint of OC, and they touch Barrow's curve at points on the lines $X = 0$ and $Y = 0$ respectively.

Two symmetrical cases of the problem were solved in antiquity:

(i) A and B lie on a diameter and

(ii) A and B are at equal distances from the centre O.

If the x-axis is the line of symmetry, in both cases $c_2 = 0$. The rectangular hyperbola degenerates into the two lines $x = c_1$ and $y = 0$, and Barrow's curve becomes the line $y = 0$ and a circle which passes through O. In case (ii) this circle passes through A and B (figure 7).

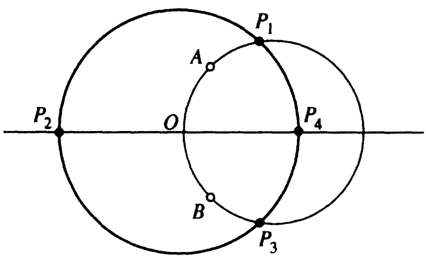

FIGURE 7.

Optical problems in general

During the same period as Huygens' and Sluse's work, James Gregory attempted an analytical solution of Alhazen's Problem but without the same success. However, in Proposition 34 of his *Optica Promota* of 1663 he states that if a ray is reflected from A to B at a surface, reflection occurs at a point where the surface touches an ellipsoid which has its foci at A and B. This is, of course, an application of the reflection property of the ellipse. In Alhazen's Problem there are up to four real confocal conics which touch the circle, and they may be ellipses or hyperbolae; the hyperbole define backward reflections.

A smooth surface can be approximated near any point by the tangent plane there; since $|AP| + |PB|$ is a minimum for reflection in a plane it follows that at a proper reflection point on any smooth surface the total distance is *stationary*. The curvature of the surface can change the minimum character, but the total distance remains 'nearly constant' for small deviations away from the reflection point. Similarly, for backward reflection from a plane the difference $|PA| - |PB|$ is a maximum, and for reflection from a general surface the difference is therefore stationary. (In the case of an ellipse or hyperbola with A and B at the foci these functions are *exactly* constant!).

This does not, however, exhaust the stationary values; on any smooth surface $|PA| + |PB|$ is stationary when APB are collinear in that order, and $|PA| - |PB|$ is stationary when the points are collinear in the order ABP or BAP. Thus in Alhazen's Problem the reflection points do not necessarily define the global extrema of these functions.

The connexion with reflection gives a particularly satisfying character to extremal problems for paths which visit given curves and surfaces. For example: find the shortest closed path which visits all three sides of a given triangle ABC. If the triangle is acute-angled, the shortest path follows a continuous light ray which is reflected at the sides [8]; if the reflection points are L, M, N then LMN is the pedal triangle which joins the feet of the perpendiculars from the vertices. But if angle A (say) is not acute then the shortest path is the altitude from A to BC and back again. So what is the significance of the pedal triangle in this case? The corresponding problem for a quadrilateral is even more interesting, and we will not spoil the reader's enjoyment by saying more about it.

Heron's observation that the reflected ray takes the shortest distance was elevated by Fermat to a general optical principle of least time, from which he derived the law of refraction. If a light ray passes from a point A in one medium to a point B in a different medium and the separating surface is smooth, the minimum time is achieved when the ray is refracted at the surface in such a way that the incident and refracted rays and the normal are coplanar, and the sines of the angles of incidence and refraction are proportional to the speeds of light in the two media. This is often presented as a calculus exercise in the form

> A life-guard is standing well back on the beach and along the coast sights a child in trouble in the sea. If he runs and swims at given rates, where should he enter the sea to reach the child as quickly as possible?

The problem can in fact be solved without calculus by 'Ptolemy's inequality', as shown in [6] and [8].

In general Fermat's principle only demands a *stationary* value of the time; nevertheless, any two points sufficiently close on a path do satisfy the minimum condition. The idea generalizes to the 'principle of least action', which is an astonishingly economical way to formulate all fundamental physical laws [4]. It is satisfying that classical mechanics and the classical limit of wave mechanics can both be interpreted in terms of a ray or wave passing through a refracting medium of non-uniform density.

Bibliography

The reason for writing this rather cursory review is that in spite of his importance al-Haytham is not a figure popularly known in the West, even among scientists and mathematicians, and Alhazen's Problem (as well as other optical problems) deserve to be known to every new generation. The following references, to which I am heavily indebted, should be useful for further study.

The article by A.I. Sabra in [9] has an account of al-Haytham's achievements in science and mathematics, with a complete bibliography of his writings; the same author's book [10] describes the quest for the law of refraction and al-Haytham's part in this. His experimental technique is described in fascinating detail in Omar's book [7] on the *Kitab*.

For the optical problem Lohne's article [5] is the best reference. It contains a comprehensive history, well illustrated with diagrams, and references to the prime sources.

Bode [2] has further accounts of work on the problem between the seventeenth and nineteenth centuries. Other references which may be useful are Baker [1] and Bruins [3]. For al-Haytham's work on the geometrical postulates there is a modern edition [11] of his commentary on Euclid's Elements.

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[2] can be found at the British Library and [7] at the Oriental Institute, Oxford; the remaining books and articles are obtainable at the Radcliffe Science Library, Oxford.

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